Scientific and Social Research, 2025, Volume 7, Issue 4
https://ojs.bbwpublisher.com/index.php/SSR
Online ISSN: 2981-9946
Print ISSN: 2661-4332

A New Proof of the (H.2) Supercongruence of Van Hamme by a WZ Pair

Feng-Xin Ding*

College of Science, Tianjin University of Technology, Tianjin 300384, China

**Author to whom correspondence should be addressed.*

Abstract: This paper presents a new proof of Van Hamme's supercongruence (H.2) using the WZ method, combining transformation and summation of WZ pairs and the properties of the p-adic Gamma function. Moreover, under the condition $p \equiv 3 \pmod 4$, the study proves a supercongruence modulo p^3.

Keywords: WZ pair; Supercongruence; p-adic Gamma function

Online publication: May 2, 2025

1. Introduction

In 1997, Van Hamme proposed 13 conjectures of Ramanujan-type supercongruences, denoted as (A.2)–(M.2) [1]. For example, he proposed the conjecture (H.2):

$$\sum_{n=0}^{\frac{p-1}{2}} \frac{(1/2)_n^3}{(1)_n^3} \equiv \begin{cases} -\Gamma_p(1/4)^4 & \left(\bmod\ p^2\right), \quad p \equiv 1 \pmod 4, \\ 0 & \left(\bmod\ p^2\right), \quad p \equiv 3 \pmod 4, \end{cases} \tag{H.2}$$

where p is an odd prime, $\Gamma p(x)$ is the p-adic Gamma function, and the Pochhammer symbol is defined as [2–3]:

$$(s)_n = \begin{cases} s(s+1)\cdots(s+n-1), & n \in \mathbb{Z}^+, \\ 1, & n = 0. \end{cases}$$

Focusing on these conjectures, Van Hamme and numerous researchers conducted extensive investigations, using methods such as the WZ (Wilf-Zeilberger) method, hypergeometric series transformation formulas, and p-adic analysis, to provide proofs of those supercongruences [2, 4]. Complete details can be found in the references [5–11]. Subsequently, researchers extended these supercongruences and proposed multiple generalizations, thereby further enriching the research content in this field. For details, see references [12–15].

Introduced by Wilf and Zeilberger in 1990, the WZ method provides a systematic approach to verifying combinatorial identities [4]. $F(n,k)$ is hypergeometric if both ratios $F(n+1, k)/(F(n, k)$ and $F(n,k+1)/F(n,k)$ are rational functions in n and k [16]. A pair of hypergeometric functions ($F(n,k)$, $G(n,k)$) is called a WZ pair if they satisfy the recurrence relation [17]:

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k).$$

In addition, for the WZ pair ($F(n,k)$, $G(n,k)$), define

$$\tilde{F}(n,k)=G(n,-k), \qquad \tilde{G}(n,k)=F(n,-k),$$

then

$$\tilde{F}(n,k-1)-\tilde{F}(n,k)=\tilde{G}(n+1,k)-\tilde{G}(n,k). \tag{1.1}$$

Employing WZ pairs along with the recurrence relation (1.1), Zudilin proved many Ramanujan-type supercongruences, and later in 2016, Osburn-Zudilin established a proof of Van Hamme's (K.2) [9, 18]. Since no suitable WZ pair for (H.2) was found, (H.2) was not proved by the WZ method.

Motivated by their work, the authors give a WZ pair and derive a new proof of supercongruence (H.2) with the p-adic Gamma function. Moreover, under the condition $p \equiv 3 \pmod 4$, the authors re-prove a supercongruence modulo p^3, which is the following theorem:

Theorem 1.1[6]. Let p be an odd prime, then [6]:

$$\sum_{n=0}^{\frac{p-1}{2}} \frac{(1/2)_n^3}{(1)_n^3} \equiv -\frac{p^2}{16}\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3} \quad if\ p \equiv 3 \pmod 4.$$

In Section 2, the authors present some preliminary knowledge and conclusions. In Section 3, the authors give a detailed proof of supercongruence (H.2) and Theorem 1.1.

2. Preliminaries and conclusions

Let us revisit some fundamental properties of the p-adic Gamma function.

Lemma 2.1. Let p be an odd prime and $x \in \mathbb{Z}_p$, then [2]:

(i)
$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)}=\begin{cases}-x, & v_p(x)=0,\\ -1, & v_p(x)>0,\end{cases} \tag{2.1}$$

where $v_p(x)$ is the exponent of p in the decomposition of x into a product of prime powers.

(ii)
$$\frac{\Gamma_p(x+m)}{\Gamma_p(x)}=(-1)^m \prod_{\substack{0\le k<m \\ v_p(x+k)=0}} (x+k), \tag{2.2}$$

where $m \in \mathbb{Z}^+$.

(iii)
$$\Gamma_p(x)\Gamma_p(1-x)=(-1)^{a_0(x)}, \tag{2.3}$$

where $a_0(x) \in \{1, 2, \ldots, P\}$, such that $a_0(x) \equiv x$ (mod p), Now, the authors present two preliminary results for (H.2).

Lemma 2.2. Let p be an odd prime, then the authors have

$$\frac{(3/4)_{\frac{p-1}{2}}}{(1/4)_{\frac{p-1}{2}}}=\begin{cases}-\frac{2p-1}{p}\Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)\Gamma_p\left(\frac{1}{4}-\frac{p}{2}\right)\Gamma_p(1/4)^2, & p\equiv 1 \pmod 4,\\ -\frac{p(2p-1)}{4^2}\Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)\Gamma_p\left(\frac{1}{4}-\frac{p}{2}\right)\Gamma_p(1/4)^2, & p\equiv 3 \pmod 4.\end{cases}$$

Proof: Firstly, the authors consider the case $p \equiv 1 \pmod 4$.

Note that $(1/4)_{\frac{p-1}{2}}$ has exactly a multiple of p, which is $p/4$, so by (2.2),

$$\frac{(3/4)_{\frac{p-1}{2}}}{(1/4)_{\frac{p-1}{2}}}=(-1)^{\frac{p-1}{2}}\frac{\Gamma_p\left(\frac{2p+1}{4}\right)}{\Gamma_p(3/4)}(-1)^{-\frac{p-1}{2}}\frac{\Gamma_p(1/4)}{\frac{p}{4}\Gamma_p\left(\frac{2p-1}{4}\right)}$$

$$=\frac{4}{p}\frac{\Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)}{\Gamma_p\left(-\frac{1}{4}+\frac{p}{2}\right)}\frac{\Gamma_p(1/4)}{\Gamma_p(3/4)}.$$

By (2.3), the following is obtained

$$\frac{\Gamma_p(1/4)}{\Gamma_p(3/4)}=\Gamma_p\left(\frac{1}{4}\right)(-1)^{-\frac{3p+1}{4}}\Gamma_p\left(\frac{1}{4}\right)=(-1)^{-\frac{3p+1}{4}}\Gamma_p\left(\frac{1}{4}\right)^2, \tag{2.4}$$

and

$$\frac{\Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)}{\Gamma_p\left(-\frac{1}{4}+\frac{p}{2}\right)}=\Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)(-1)^{-\frac{p-1}{4}}\Gamma_p\left(\frac{5}{4}-\frac{p}{2}\right). \tag{2.5}$$

In view of (2.1), (2.4), and (2.5), the authors deduce that

$$\frac{(3/4)_{\frac{p-1}{2}}}{(1/4)_{\frac{p-1}{2}}}=(-1)^{-p}\frac{4}{p}\Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)\Gamma_p\left(\frac{5}{4}-\frac{p}{2}\right)\Gamma_p\left(\frac{1}{4}\right)^2$$

$$=-\frac{4}{p}\Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)\left(-\frac{1-2p}{4}\right)\Gamma_p\left(\frac{1}{4}-\frac{p}{2}\right)\Gamma_p\left(\frac{1}{4}\right)^2.$$

Through simple calculations, we can complete the proof for the case $p \equiv 1 \pmod 4$.

And then, the authors consider the case $p \equiv 3 \pmod 4$. Note that $(3/4)_{\frac{p-1}{2}}$ has exactly a multiple of p which is $p/4$, so it is easy to see from (2.2) that

$$\frac{(3/4)_{\frac{p-1}{2}}}{(1/4)_{\frac{p-1}{2}}}=\frac{p}{4}\frac{\Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)}{\Gamma_p\left(-\frac{1}{4}+\frac{p}{2}\right)}\frac{\Gamma_p(1/4)}{\Gamma_p(3/4)}.$$

Similar to the process above, the authors can complete the proof for this case.

Lemma 2.3. Let $p\geq 5$ be a prime, then

$$(2p-1)\frac{(1/2)_{p-1}}{(1)^2_{\frac{p-1}{2}}}\equiv p(-1)^{\frac{p-1}{2}} \pmod{p^4}.$$

Proof: Through simple calculations, it is easy to see

$$(2p-1)\frac{(1/2)_{p-1}}{(1)^2_{\frac{p-1}{2}}}=2^{-2(p-1)}p\binom{2p-1}{p-1}\binom{p-1}{\frac{p-1}{2}}.$$

The authors notice the well-known supercongruences discovered by J. Wolstenholme and F. Morley, respectively [19–20],

$$\binom{2p-1}{p-1}\equiv 1 \pmod{p^3}, \qquad \binom{p-1}{\frac{p-1}{2}}\equiv(-1)^{\frac{p-1}{2}}2^{2(p-1)} \pmod{p^3},$$

therefore,

$$(2p-1)\frac{(1/2)_{p-1}}{(1)^2_{\frac{p-1}{2}}}\equiv p(-1)^{\frac{p-1}{2}} \pmod{p^4}.$$

The subsequent result regarding the expansion of the p-adic Gamma function holds significant importance in the following proof.

Lemma 2.4 [6]. For $p\geq 5$ be a prime, $r\in\mathbb{N}$, $a\in\mathbb{Z}_p$, $m\in\mathbb{C}_p$, satisfying $v_p(m)\geq 0$ and $t\in\{0,1,2\}$, then

$$\frac{\Gamma_p(a+mp^r)}{\Gamma_p(a)}\equiv\sum_{k=0}^{t}\frac{G_k(a)}{k!}(mp^r)^k \pmod{p^{(t+1)r}},$$

where $G_k(a)=\frac{\Gamma_p^{(k)}(a)}{\Gamma_p(a)}\in\mathbb{Z}$ and $\Gamma_p^{(k)}(a)$ is the k-th derivative of $\Gamma_p(a)$.

3. A New Proof of Supercongruence (H.2) and Theorem 1.1

Now, the authors present a new proof of supercongruence (H.2) and Theorem 1.1.

When $p = 3$, the following is obtained

$$\sum_{n=0}^{1}\frac{(1/2)_n^3}{(1)_n^3}=1+\left(\frac{1}{2}\right)^3=\frac{9}{8}\equiv 0 \pmod{3^2}.$$

Since

$$\Gamma_3\left(\frac{1}{4}\right)\equiv\Gamma_3(1)\equiv -1 \pmod{3},$$

the authors obtain

$$\sum_{n=0}^{1}\frac{(1/2)_n^3}{(1)_n^3}-\left[-\frac{3^2}{16}\Gamma_3\left(\frac{1}{4}\right)^4\right]=\frac{9}{8}+\frac{9}{16}\Gamma_3\left(\frac{1}{4}\right)^4=\frac{9}{16}\left[2+\Gamma_3\left(\frac{1}{4}\right)^4\right]\equiv 0 \pmod{3^3}.$$

Therefore, (H.2) and Theorem 1.1 hold when $p = 3$.

When $p\geq5$, the authors have WZ pair ($F(n,k)$, $G(n,k)$)

$$G_1(n,k)=(-1)^k\frac{(1/2)_n^2(1/2)_{n-k}}{(1)_n^2(1)_{n+k}}\frac{(1/2)_k^2(3/4)_k}{(1/4)_k},$$

$$F_1(n,k)=\frac{(-1)^k}{2}\frac{(1/2)_n^2(1/2)_{n-k-1}}{(1)_{n-1}^2(1)_{n+k}}\frac{(1/2)_k^2(3/4)_k}{(1/4)_{k+1}}.$$

Let $\tilde{F}(n,k)=G(n,-k),\tilde{G}(n,k)=F(n,-k)$, the authors have

$$\tilde{F}(n,k)=(-1)^k\frac{(1/2)_n^2(1/2)_{n+k}}{(1)_n^2(1)_{n-k}}\frac{(3/4)_k}{(1/2)_k^2(1/4)_k},$$

$$\tilde{G}(n,k)=\frac{(-1)^{k-1}}{2}\frac{(1/2)_n^2(1/2)_{n+k-1}}{(1)_{n-1}^2(1)_{n-k}}\frac{(3/4)_{k-1}}{(1/2)_k^2(1/4)_k},$$

where $\tilde{F}(n,k)$ and $\tilde{G}(n,k)$ satisfying (1.1).

Summing (1.1) over $n=0,1,\cdots,\frac{p-1}{2}$ the authors get

$$\sum_{n=0}^{\frac{p-1}{2}}\tilde{F}(n,k-1)-\sum_{n=0}^{\frac{p-1}{2}}\tilde{F}(n,k)=\tilde{G}\left(\frac{p+1}{2},k\right)-\tilde{G}(0,k)=\tilde{G}\left(\frac{p+1}{2},k\right). \tag{3.1}$$

Furthermore, for $k=1,2,\cdots,\frac{p-1}{2}$, the authors have

$$\tilde{G}\left(\frac{p+1}{2},k\right)=\frac{(-1)^{k-1}}{2}\frac{(1/2)_{\frac{p+1}{2}}^2(1/2)_{\frac{p-1}{2}+k}}{(1)_{\frac{p-1}{2}}^2(1)_{\frac{p+1}{2}-k}}\frac{(3/4)_{k-1}}{(1/2)_k^2(1/4)_k}$$

$$=\frac{(-1)^{k-1}}{2}\left(\frac{p}{2}\right)^3\frac{(1/2)_{\frac{p-1}{2}}^3(\frac{p}{2}+1)_{k-1}(3/4)_{k-1}}{(1)_{\frac{p-1}{2}}^2(1)_{\frac{p+1}{2}-k}(1/2)_k^2(1/4)_k}.$$

It is easy to see that $(1)^2_{\frac{p-1}{2}}(1)_{\frac{p+1}{2}-k}(1/2)^2_k$ is coprime to p. And then, when $p \equiv 1$ (*mod* 4), $(1/4)_k$ has a multiple of p, which is $p/4$ when $\frac{p+3}{4} \le k \le \frac{p-1}{2}$. When $p \equiv 3$ (mod 4), $(1/4)_k$ is coprime to p. Therefore, the authors can conclude

$$\tilde{G}\left(\frac{p+1}{2},k\right) \equiv \begin{cases} 0 \pmod{p^2}, & p \equiv 1 \pmod 4, \\ 0 \pmod{p^3}, & p \equiv 3 \pmod 4. \end{cases}$$

Comparing this result with (3.1), noticing that $\frac{1}{(1)_{n-\frac{p-1}{2}}} = 0$ while $n < \frac{p-1}{2}$, the authors see that

$$\sum_{n=0}^{\frac{p-1}{2}} \tilde{F}(n,0) \equiv \sum_{n=0}^{\frac{p-1}{2}} \tilde{F}\left(n,\frac{p-1}{2}\right) \tag{3.2}$$

$$\equiv \begin{cases} \tilde{F}\left(n,\frac{p-1}{2}\right) \pmod{p^2}, & p \equiv 1 \pmod 4, \\ \tilde{F}\left(n,\frac{p-1}{2}\right) \pmod{p^3}, & p \equiv 3 \pmod 4. \end{cases}$$

By Lemma 2.2, the authors get

$$\tilde{F}\left(\frac{p-1}{2},\frac{p-1}{2}\right) = (-1)^{\frac{p-1}{2}} \frac{(1/2)_{p-1}}{(1)^2_{\frac{p-1}{2}}} \frac{(3/4)_{\frac{p-1}{2}}}{(1/4)_{\frac{p-1}{2}}} \tag{3.3}$$

$$= \begin{cases} -\frac{2p-1}{p} \frac{(1/2)_{p-1}}{(1)^2_{\frac{p-1}{2}}} \Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)\Gamma_p\left(\frac{1}{4}-\frac{p}{2}\right)\Gamma_p\left(\frac{1}{4}\right)^2, & p \equiv 1 \pmod 4, \\ \frac{p(2p-1)}{4^2} \frac{(1/2)_{p-1}}{(1)^2_{\frac{p-1}{2}}} \Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)\Gamma_p\left(\frac{1}{4}-\frac{p}{2}\right)\Gamma_p\left(\frac{1}{4}\right)^2, & p \equiv 3 \pmod 4. \end{cases}$$

The authors now use Lemma 2.4 to obtain

$$\Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right) \equiv \Gamma_p\left(\frac{1}{4}\right)\left(1+\frac{p}{2}G_1\left(\frac{1}{4}\right)\right) \pmod{p^2},$$

and

$$\Gamma_p\left(\frac{1}{4}-\frac{p}{2}\right) \equiv \Gamma_p\left(\frac{1}{4}\right)\left(1-\frac{p}{2}G_1\left(\frac{1}{4}\right)\right) \pmod{p^2},$$

then have

$$\Gamma_p\left(\frac{1}{4}+\frac{p}{2}\right)\Gamma_p\left(\frac{1}{4}-\frac{p}{2}\right)\Gamma_p\left(\frac{1}{4}\right)^2 \equiv \Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^2}. \tag{3.4}$$

Using Lemma 2.3, in view of (3.3) and (3.4), the authors have

$$\sum_{n=0}^{\frac{p-1}{2}}\frac{(1/2)_n^3}{(1)_n^3}\equiv\begin{cases}-\Gamma_p\left(\frac{1}{4}\right)^4 & \left(\text{mod } p^2\right), \quad p\equiv 1\left(\text{mod } 4\right),\\ -\frac{p^2}{16}\Gamma_p\left(\frac{1}{4}\right)^4 & \left(\text{mod } p^3\right), \quad p\equiv 3\left(\text{mod } 4\right).\end{cases}$$

The authors completed the proof.

In 2016, Long and Ramakrishna expanded the the p-adic Gamma functions into their Taylor series expansions to prove the supercongruence modulo p^3 [6]:

$$\sum_{n=0}^{\frac{p-1}{2}}\frac{(1/2)_n^3}{(1)_n^3}\equiv\begin{cases}-\Gamma_p\left(\frac{1}{4}\right)^4 & \left(\text{mod } p^3\right), \quad p\equiv 1\left(\text{mod } 4\right),\\ -\frac{p^2}{16}\Gamma_p\left(\frac{1}{4}\right)^4 & \left(\text{mod } p^3\right), \quad p\equiv 3\left(\text{mod } 4\right).\end{cases} \tag{3.5}$$

Notice that Theorem 1.1 corresponds to the $p \equiv 3 \pmod 4$ case in (3.5). Unlike the approach proposed by Long-Ramakrishna, our proof of Theorem 1.1 avoids complex hypergeometric transformations through the WZ method.

Disclosure statement

The author declares no conflict of interest.

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