Scientific and Social Research, 2025, Volume 7, Issue 5
https://ojs.bbwpublisher.com/index.php/SSR
Online ISSN: 2981-9946
Print ISSN: 2661-4332

Optimal Control Study of an SEIQRS Model with Three Interventions on Complex Networks

Shangqing Gong[1,2], **Jianyong Dai**[1,2]*

[1]School of Resources Environment and Safety Engineering, University of South China, Hengyang 421001, Hunan, China
[2]Key Laboratory of Emergency Safety Technology and Equipment of Nuclear Facilities in Hunan Province, Hengyang 421001, Hunan, China

**Author to whom correspondence should be addressed.*

Abstract: Infectious diseases pose a significant threat to human life, health, and safety. Therefore, it is crucial to develop effective control strategies. This paper aims to address this concern through the construction of an SEIQRS model on complex networks. This model focuses on viruses that have an incubation period and are infectious during this period. In order to minimize the costs, optimal control theory is used to solve the time-varying control problem of vaccination, quarantine, and treatment. Subsequently, numerical simulations are performed to analyze the pros and cons of different control combinations, as well as the impact of parameters on the effectiveness of control. By doing so, better control strategies can be developed, and the relationship between parameters, contagion, and control can be revealed.

Keywords: Optimal control; Complex networks; Epidemic model; Numerical simulation

Online publication: June 6, 2025

1. Introduction

The world is currently facing a serious situation with regard to the spread of infectious diseases. There is therefore an urgent need to study the patterns and conditions of transmission of these diseases. On the one hand, there is the re-emergence of infectious diseases that used to be widespread in the past. On the other hand, new infectious diseases such as COVID-19 are emerging from time to time, threatening the physical and mental health of the people and the public safety of society. Taking steps to block the spread of the virus is crucial in protecting people's lives and health, as well as maintaining social stability.

Classical infectious disease dynamics revealed the transmission threshold law through the SIR/SIS hamartomatous model, but assumed a limitation of uniform population contact [1–3]. Complex network theory breaks through this limitation by abstracting individual exposure to network structure, and Pastor-Satorras showed that the transmission threshold in scale-free networks tends to zero as the network size grows, which explains the mechanism of the sustained global spread of modern infectious diseases [4]. Current research focuses on: the effect of network topological features (degree distribution, clustering coefficients) on spread, the spreading patterns of dynamic and weighted networks, and the modeling of factors such as media awareness and protective behavior [3, 5–7]. Control strategies have evolved from constant value control to optimal control, and balancing the cost and effectiveness of prevention and control by means of vaccination and isolation treatment has formed an important research direction [8–9]. This field provides theoretical support for the development of precise prevention and control strategies by constructing a networked propagation model.

An attempt is made in this paper to analyze the advantages and disadvantages of different control combinations to provide some theoretical basis for the development of control strategies. The purpose of this investigation is to examine the effect of different control combinations and the effect of different model parameters on the control effectiveness.

2. Dynamic optimal control of an SEIQRS epidemic model on a complex network

2.1. SEIQRS epidemic model

The SEIR model divides the total population into four compartments: Susceptible (S) (uninfected individuals at risk of infection), Exposed (E) (infected individuals in the latent period with some infectiousness but no symptoms), Infected (I) (symptomatic individuals with higher infectiousness), and Recovered (R) (immune individuals no longer participating in viral transmission). By integrating the mean-field theory of complex networks, and assuming a constant total population, a latent period with infectiousness, and bilinear incidence rates, the SEIRS model on complex networks can be formulated as follows:

$$\begin{aligned}
\frac{dS_k(t)}{dt} &= -\beta k S_k(t)(\Theta_1(t)+\theta\Theta_2(t))+\delta R_k(t) \\
\frac{dE_k(t)}{dt} &= \beta k S_k(t)(\Theta_1(t)+\theta\Theta_2(t))-\eta E_k(t) \\
\frac{dI_k(t)}{dt} &= \eta E_k(t)-\mu I_k(t) \\
\frac{dR_k(t)}{dt} &= \mu I_k(t)-\delta R_k(t)
\end{aligned} \tag{1}$$

In Equation 1, β is the effective transmission rate, θ is the coefficient of the effective transmission rate of the incubator, δ is the rate of immune failure, k is the degree of the node, η is the outbreak rate, and μ is the natural recovery rate.

$\Theta_1(t)=\frac{1}{\langle k\rangle}\sum_k kP_kI_k(t)$ and $\Theta_2(t)=\frac{1}{\langle k\rangle}\sum_k kP_kE_k(t)$ represent the average contact rates, which denote the probability that any node is connected to an infected individual (I) or an exposed individual (E). The average degree of the node is denoted by $<k>=\sum_k kP_k$, and $P_k(k=1,2,\ldots,n)$ represents the degree distribution of the node.

Additionally, three controls, namely vaccination $\alpha(t)S_k(t)$, quarantine $q(t)I_k(t)$, and treatment $\sigma(t)I_k(t)$, are introduced to the SEIRS model on complex networks to obtain the SEIQRS model with controls on complex networks (Equation 2). Here, the parameters α, q, and σ represent the proportion of the population receiving vaccination, quarantine, and treatment, respectively.

$$
\begin{aligned}
\frac{dS_k(t)}{dt} &= -\beta k S_k(t)(\Theta_1(t)+\theta\Theta_2(t)) - \alpha S_k(t) + \delta R_k(t) \\
\frac{dE_k(t)}{dt} &= \beta k S_k(t)(\Theta_1(t)+\theta\Theta_2(t)) - \eta E_k(t) \\
\frac{dI_k(t)}{dt} &= \eta E_k(t) - (\mu + q + \sigma) I_k(t) \\
\frac{dQ_k(t)}{dt} &= q I_k(t) - \varepsilon Q_k(t) \\
\frac{dR_k(t)}{dt} &= (\mu+\sigma) I_k(t) + \varepsilon Q_k(t) + \alpha S_k(t) - \delta R_k(t)
\end{aligned} \tag{2}
$$

where α is the effective vaccination rate, q is the isolation rate, and σ is the treatment recovery rate. Q(t) represents the proportion of the population under quarantine as a function of time. Q is based on the assumption that the isolatee de-isolates and resumes the original social relationship (the original connecting edge) immediately after recovery. The initial conditions of model (2) are:

$$
\begin{cases} 0 \le S_k(0), E_k(0), I_k(0), Q_k(0), R_k(0) \le 1 \\ S_k(0) + E_k(0) + I_k(0) + Q_k(0) + R_k(0) = 1 \end{cases}, (k = 1,2,\ldots,n) \tag{3}
$$

2.2. Optimal controls

In this section, the optimal control theory will be used to discuss the optimal control problem for each control combination and compare them.

The control variables u(t) and the Lagrangian function L(x(t), u(t)) for different control combinations are shown in **Table 1**, where α(t), q(t), and σ(t) are defined as $\alpha(t)=(\alpha_1(t),\alpha_2(t), ..., \alpha_n(t))$, $q(t)=(q_1(t), q_2(t), ..., q_n(t))$, $\sigma(t)=(\sigma_1(t), \sigma_2(t), ..., \sigma_n(t))$. Define the control set $U=\{u(t), 0 \le \alpha_k(t), q_k(t), \sigma_k(t) \le 1, t \in [0,T]\}$ with T > 0 as the target moment and construct the objective function $J(u) = \int_0^T L(u(t))\, dt$. The meaning of the objective function is the cost of the control measures, including the cost of implementing the control and the cost of the reduced social benefits resulting from contracting the disease, where the cost of the reduced social benefits resulting from contracting the disease is characterized by the number of infected persons. It is assumed that both types of costs are equally important and have the same weighting factor. The optimal control problem for model (2) under the seventh control combination is as follows:

Solving the optimal control problem involves finding an optimal control u* such that $J(u) = min_{u(t) \in u} J(u)$. For equation 2, there are seven combinations of vaccination, quarantine, and treatment. The optimal control problems under the other six control combinations can be considered as special cases when some of the control parameters of combination 7 are constants of zero.

Table 1. Control variables and Lagrangian functions ∈

Combination	Control parameter	u(t)	L(x(t), u(t))
1	α	$\alpha(t)$	$\sum_{k=1}^{n}[I_k(t)+\frac{1}{2}A_k\alpha_k^2(t)]$
2	q	$q(t)$	$\sum_{k=1}^{n}[I_k(t)+\frac{1}{2}B_kq_k^2(t)]$
3	σ	$\sigma(t)$	$\sum_{k=1}^{n}[I_k(t)+\frac{1}{2}C_k\sigma_k^2(t)]$
4	α,q	$(\alpha(t),q(t))$	$\sum_{k=1}^{n}[I_k(t)+\frac{1}{2}A_k\alpha_k^2(t)+\frac{1}{2}B_kq_k^2(t)]$
5	α,σ	$(\alpha(t),\sigma(t))$	$\sum_{k=1}^{n}[I_k(t)+\frac{1}{2}A_k\alpha_k^2(t)+\frac{1}{2}C_k\sigma_k^2(t)]$
6	q,σ	$(q(t),\sigma(t))$	$\sum_{k=1}^{n}[I_k(t)+\frac{1}{2}B_kq_k^2(t)+\frac{1}{2}C_k\sigma_k^2(t)]$
7	α,q,σ	$(\alpha(t),q(t),\sigma(t))$	$\sum_{k=1}^{n}[I_k(t)+\frac{1}{2}A_k\alpha_k^2(t)+\frac{1}{2}B_kq_k^2(t)+\frac{1}{2}C_k\sigma_k^2(t)]$

To solve the optimal control problem (Equation 2) for the combination 7 as an example, the Hamiltonian function (Equation 4) needs to be defined first.

$$H = L(x(t),u(t)) + \sum_{k=1}^{n}[\lambda_{1k}(t)\dot{S}_k(t) + \lambda_{2k}(t)\dot{E}_k(t) + \lambda_{3k}(t)\dot{I}_k(t) + \lambda_{4k}(t)\dot{Q}_k(t) + \lambda_{5k}(t)\dot{R}_k(t)] \tag{4}$$

The adjoint variables, denoted as $\lambda_{1k}(t)$, $\lambda_{2k}(t)$, $\lambda_{3k}(t)$, $\lambda_{4k}(t)$, $\lambda_{5k}(t)$, are relevant for this function. According to Pontryagin's principle of great value, the optimal solution, denoted as O^* can be obtained. In this case, $O^*=\{S^*_1(t), E^*_1(t), I^*_1(t), Q^*_1(t), R^*_1(t),..., Q^*_n(t), R^*_n(t),\}$ is the solution of Equation 2 in the state of optimal control u*(t). Additionally, the optimal solution of the control variables can be determined by Equation 6, provided that the conditions $\Theta_1^*(t) = \frac{1}{\langle k \rangle}\sum_k kP_kI_k^*(t)$, $\Theta_2^*(t) = \frac{1}{\langle k \rangle}\sum_k kP_kE_k^*(t)$, and the boundary conditions (Equation 5) are satisfied. The proof of this conclusion is presented below.

$$\lambda_{1k}(T) = \lambda_{2k}(T) = \lambda_{3k}(T) = \lambda_{4k}(T) = \lambda_{5k}(T) = 0, k = 1,2,...,n \tag{5}$$

$$\begin{aligned} \alpha_k^* &= \min\{\max\{0, \frac{S_k^*(t)(\lambda_{1k}(t)-\lambda_{5k}(t))}{A_k}\},1\} \\ q_k^* &= \min\{\max\{0, \frac{I_k^*(t)(\lambda_{3k}(t)-\lambda_{4k}(t))}{B_k}\},1\} \\ \sigma_k^* &= \min\{\max\{0, \frac{I_k^*(t)(\lambda_{3k}(t)-\lambda_{5k}(t))}{C_k}\},1\} \end{aligned} \tag{6}$$

PROOF: Following Pontryagin's principle of extreme value, the Hamiltonian function (Equation 4) is defined. The adjoint variables can be calculated by the system of differential equations (Equation 7).

$$\dot{\lambda}_{1k}(t) = -\frac{\partial H}{\partial S_k}\bigg|_{O^*(t)}, \dot{\lambda}_{2k}(t) = -\frac{\partial H}{\partial E_k}\bigg|_{O^*(t)}, \dot{\lambda}_{3k}(t) = -\frac{\partial H}{\partial I_k}\bigg|_{O^*(t)} \tag{7}$$

$$\dot{\lambda}_{4k}(t) = -\frac{\partial H}{\partial Q_k}\bigg|_{O^*(t)}, \dot{\lambda}_{5k}(t) = -\frac{\partial H}{\partial R_k}\bigg|_{O^*(t)}$$

$$\frac{\partial H}{\partial \alpha_k}\bigg|_{O^*(t)} = \frac{\partial H}{\partial q_k}\bigg|_{O^*(t)} = \frac{\partial H}{\partial \sigma_k}\bigg|_{O^*(t)} = 0 \tag{8}$$

$$\dot{\lambda}_{1k} = -\frac{\partial H}{\partial S_k} = \beta k(\Theta_1(t) + \theta\Theta_2(t))(\lambda_{1k}(t) - \lambda_{2k}(t)) + \alpha_k(t)(\lambda_{1k}(t) - \lambda_{5k}(t)) \tag{9}$$

$$\dot{\lambda}_{2k} = -\frac{\partial H}{\partial E_k} = \frac{1}{<k>}\beta\theta k P_k \sum_{k=1}^{n}[kS_k(\lambda_{1k}(t) - \lambda_{2k}(t))] + \eta(\lambda_{2k}(t) - \lambda_{3k}(t))$$

$$\dot{\lambda}_{3k} = -\frac{\partial H}{\partial I_k} = -1 + \frac{1}{<k>}\beta k P_k \sum_{k=1}^{n}[kS_k(\lambda_{1k}(t) - \lambda_{2k}(t))] + q_k(t)(\lambda_{3k}(t) - \lambda_{4k}(t)) + (\mu + \sigma_k(t))(\lambda_{3k}(t) - \lambda_{5k}(t))$$

$$\dot{\lambda}_{4k} = -\frac{\partial H}{\partial Q_k} = \varepsilon(\lambda_{4k}(t) - \lambda_{5k}(t))$$

$$\dot{\lambda}_{5k} = -\frac{\partial H}{\partial R_k} = \delta(\lambda_{5k}(t) - \lambda_{1k}(t))$$

The system of differential equations (Equation 7), which is satisfied by the adjoint variables, can be derived using the boundary conditions (Equation 5) and the control equations (Equation 8). Finally, the optimal solutions of the control can be obtained as Equation 10, Equation 11, and Equation 12, respectively.

$$\frac{\partial H}{\partial \alpha_k}\bigg|_{O^*(t)} = A_k\alpha_k^*(t) - \lambda_{1k}(t)S_k^*(t) + \lambda_{5k}(t)S_k^*(t) = 0 \tag{10}$$

$$\alpha^* = \min\{\max\{0, \frac{S_k^*(t)(\lambda_{1k}(t) - \lambda_{5k}(t))}{A_k}\}, 1\}$$

$$\frac{\partial H}{\partial q_k}\bigg|_{O^*(t)} = B_k q_k^*(t) - \lambda_{3k}(t)I_k^*(t) + \lambda_{4k}(t)I_k^*(t) = 0 \tag{11}$$

$$q^* = \min\{\max\{0, \frac{I_k^*(t)(\lambda_{3k}(t) - \lambda_{4k}(t))}{B_k}\}, 1\}$$

$$\frac{\partial H}{\partial \sigma_k}\bigg|_{O^*(t)} = C_k\sigma_k^*(t) - \lambda_{3k}(t)I_k^*(t) + \lambda_{5k}(t)I_k^*(t) = 0 \tag{12}$$

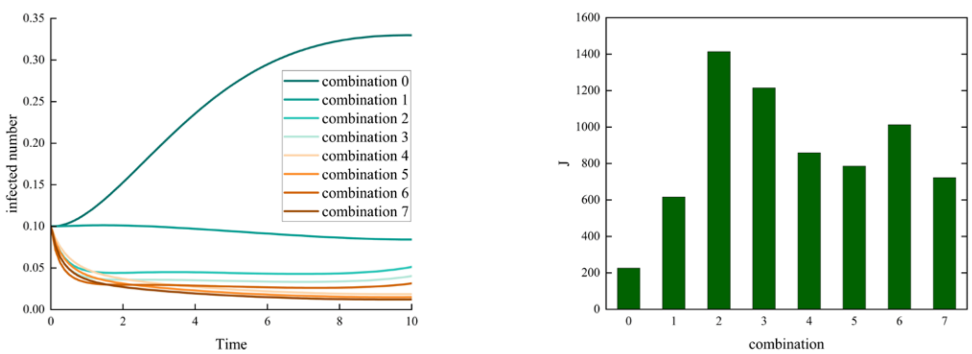
$$\sigma^* = \min\{\max\{0, \frac{I_k^*(t)(\lambda_{3k}(t) - \lambda_{5k}(t))}{C_k}\}, 1\}$$

3. Numerical simulation

According to the complex network theory, the structure of a social network aligns with that of a scale-free network. The model is assumed to be based on a BA scale-free network with the nodes' degree distribution following a power law, constructed using an evolutionary generation approach. The degree distribution of the network is $P_k=ck^{-\gamma}$, where c=11.6, γ=2.64, $\sum_{k=1}^{n}P_k=1$, n=188, $<k>$=6.9.

This section analyzes the advantages and disadvantages of different control combinations for fixed parameters. Set some parameters in the model to β=0.5, δ=0.01, μ=0.01, ε=0.35, η=0.07, θ=0.5. The initial conditions are $\sum_{k=1}^{n}S_k(0)=0.9$, $\sum_{k=1}^{n}E_k(0)=0$, $\sum_{k=1}^{n}I_k(0)=0.1$, $\sum_{k=1}^{n}Q_k(0)=0$, $\sum_{k=1}^{n}R_k(0)=0$, with the initial number of infections being 10% of the total number of randomly selected individuals. Since randomly selecting initial infected individuals could lead to varying simulation results each time, the average of 100 simulation outcomes is taken as the final result. Set the cost weight coefficients in the objective function to A_k=0.05, B_k=0.8, C_k=0.5, and the objective time is set to T=10. The control intensity of the small compartment of each degree is averaged as the control intensity of the large compartment, like $\alpha(t)=\sum_{k=1}^{n}\alpha_k(t)/n$, $q(t)=\sum_{k=1}^{n}q_k(t)/n$, $\sigma(t)=\sum_{k=1}^{n}\sigma_k(t)/n$.

The numerical simulation results are shown in **Figure 2**, where combination 0 is the case without controls. The infected number demonstrates the effects of control measures. As shown in **Figure 2(a)**, vaccination alone can only keep the number of infected individuals at its initial level, while also maintaining a gradual decrease in the infection count over time. The effectiveness difference between quarantine and treatment, whether used alone or in combination, is not significant. In combinations with vaccination, the infected number consistently decreased, while in combinations without vaccination, it rebounded after a period of reduction. Mechanistically, vaccination takes longer to work and is expected to completely eliminate infectious diseases. Quarantine and treatment reduce the infected people for a short period of time, but only work on infected people and are difficult to completely eliminate infectious diseases.



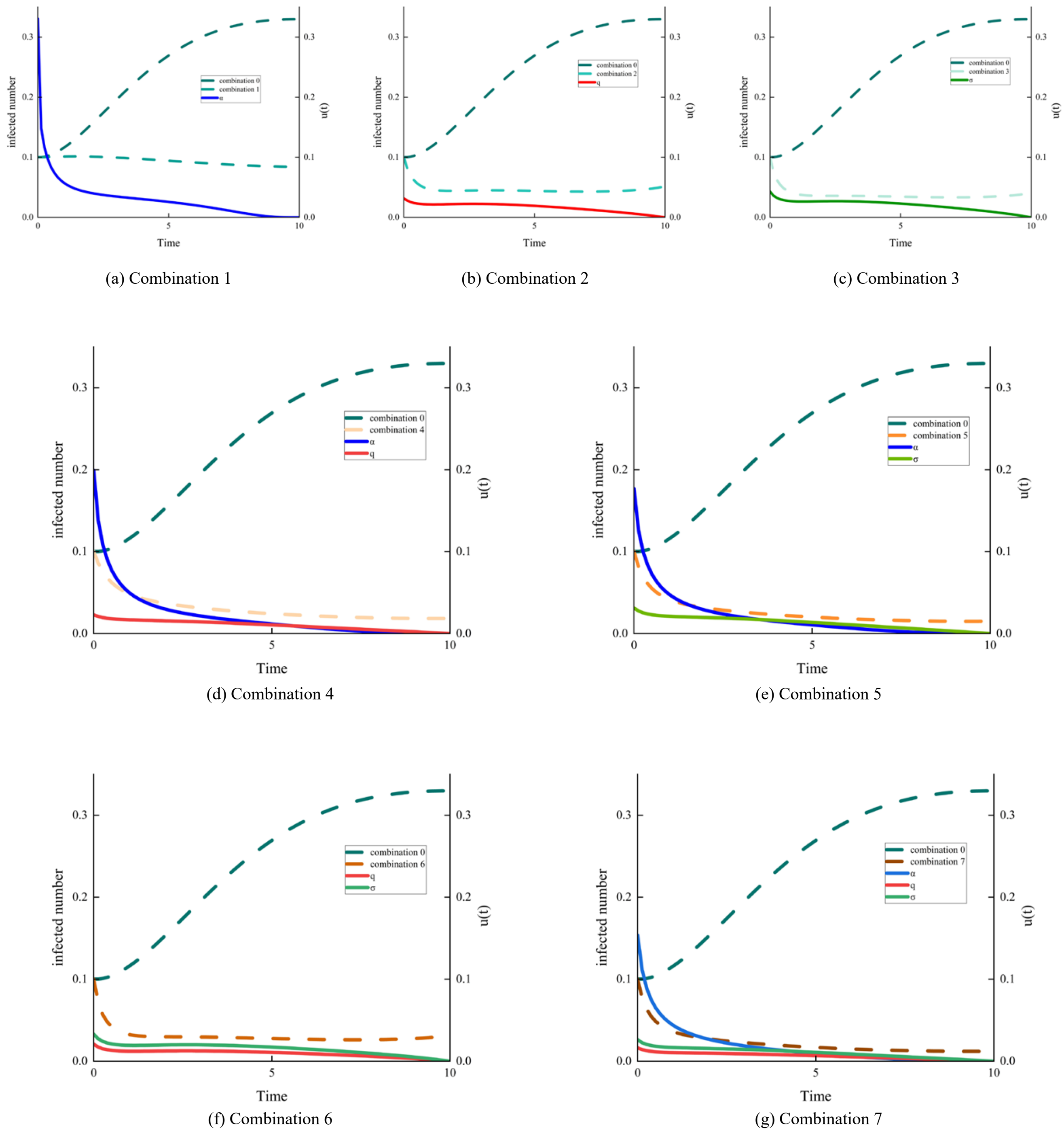
(a) Effects of different control combinations

(b) Cost of different control combinations

Figure 2. Effects and cost of different control combinations.

As shown in **Figure 2(b)**, vaccination alone is cheaper but less effective. The cost of combinations with vaccination is less than the cost of combinations without vaccination. The cost of combined control combinations is less than the cost of quarantine or treatment only, where the cost of treatment is less than the cost of quarantine.

The optimal control is shown in **Figure 3**. Vaccination is significantly higher than quarantine and treatment. Vaccination is reduced sharply at the beginning and then gradually reduced. Quarantine and treatment fluctuate in the absence of vaccination, but level off and decrease in the presence of vaccination. The combination of three controls is stronger than the combination of two controls, which is stronger than a single control.



(a) Combination 1

(b) Combination 2

(c) Combination 3

(d) Combination 4

(e) Combination 5

(f) Combination 6

(g) Combination 7

Figure 3. The infected number and the solution for optimal control

4. Conclusion

This paper analyzes the optimal control problem for SEIQRS models on complex networks. The three optimal controls of vaccination, quarantine, and treatment were solved. The effects of parameters on the effectiveness of control were studied. Theoretical results were corroborated with numerical simulations. The effects and costs of different combinations of controls were compared. From these analyses, the following conclusions were drawn.

Quarantine and treatment are more effective than vaccination for short periods of time. The effectiveness of quarantine and treatment, whether used alone or in combination, does not significantly differ. On the other hand, vaccination is effective in limiting the spread of infectious diseases and eliminating them altogether. However, vaccination alone is not effective in reducing the number of infected individuals for a short period of time.

The cost of control combinations is generally lower than the cost of implementing a single control. Combinations that involve vaccination have lower costs compared to those without vaccination. Moreover, the overall cost of quarantine and treatment is higher than the cost of vaccination.

Short-term vaccination is preferable over quarantine and treatment. Combinations involving vaccination exhibit smoother changes compared to combinations without vaccination.

Disclosure statement

The authors declare no conflict of interest.

References

[1] Kermack WO, McKendrick AG, 1927, A Contribution to the Mathematical Theory of Epidemics. Proceedings of the Royal Society of London, The Royal Society, 115(772): 700–721.

[2] Kermack WO, McKendrick AG, 1932, Contribution to the Mathematical Theory of Epidemics II: The Problem of Endemicity. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences,1932(138A): 55–83.

[3] Watts D, Strogatz S, 1998, Collective Dynamics of "Small-world" Networks. Nature, 393(6684): 440–442.

[4] Pastor-Satorras R, Castellano C, Van Mieghem P, et al., 2015, Epidemic Processes in Complex Networks. Reviews of Modern Physics, 87(3): 925–986.

[5] Liu G, Liu L, Jin Z, 2018, Dynamics Analysis of Epidemic and Information Spreading in Overlay Networks. Journal of Theoretical Biology, 2018(444): 28–37.

[6] Holme P, Saramaki J, 2012, Temporal Networks. Physics Reports, 2012(519): 97–125.

[7] Xia CY, Wang ZS, Zheng CY, et al., 2019, A New Coupled Disease-awareness Spreading Model with Mass Media on Multiplex Networks. Information Sciences, 2019(471): 185–120.

[8] Jia N, Ding L, Liu YJ, et al., 2018, Global Stability and Optimal Control of Epidemic Spreading on Multiplex Networks with Nonlinear Mutual Interaction. Physica A, 2018(502): 93–105.

[9] Fleming WH, Rishel RW, 1975, Deterministic and Stochastic Optimal Control. Springer Verlag, New York, 26–99.

Publisher's note

Bio-Byword Scientific Publishing remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.